

# On a class of $(\delta + \alpha u^2)$ -constacyclic codes over $\mathbb{F}_q[u]/\langle u^4 \rangle$

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## Abstract

Let  $\mathbb{F}_q$  be a finite field of cardinality  $q$ ,  $R = \mathbb{F}_q[u]/\langle u^4 \rangle = \mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q + u^3\mathbb{F}_q$  ( $u^4 = 0$ ) which is a finite chain ring, and  $n$  be a positive integer satisfying  $\gcd(q, n) = 1$ . For any  $\delta, \alpha \in \mathbb{F}_q^\times$ , an explicit representation for all distinct  $(\delta + \alpha u^2)$ -constacyclic codes over  $R$  of length  $n$  is given, and the dual code for each of these codes is determined. For the case of  $q = 2^m$  and  $\delta = 1$ , all self-dual  $(1 + \alpha u^2)$ -constacyclic codes over  $R$  of odd length  $n$  are provided.

*Keywords:* Constacyclic code; Dual code; Self-dual code; Finite chain ring

**Mathematics Subject Classification (2000)** 94B15, 94B05, 11T71

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## 1. Introduction

Algebraic coding theory deals with the design of error-correcting and error-detecting codes for the reliable transmission of information across noisy channel. The class of constacyclic codes play a very significant role in the theory of error-correcting codes.

Let  $\Gamma$  be a commutative finite ring with identity  $1 \neq 0$ , and  $\Gamma^\times$  be the multiplicative group of units of  $\Gamma$ . For any  $a \in \Gamma$ , we denote the ideal of  $\Gamma$  generated by  $a$  as  $\langle a \rangle_\Gamma$ , or  $\langle a \rangle$  for simplicity, i.e.,  $\langle a \rangle_\Gamma = a\Gamma = \{ab \mid b \in \Gamma\}$ . For any ideal  $I$  of  $\Gamma$ , we will identify the element  $a + I$  of the residue class ring  $\Gamma/I$  with  $a \pmod{I}$  for any  $a \in \Gamma$  in this paper.

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A *code* over  $\Gamma$  of length  $N$  is a nonempty subset  $\mathcal{C}$  of  $\Gamma^N = \{(a_0, a_1, \dots, a_{N-1}) \mid a_j \in \Gamma, j = 0, 1, \dots, N-1\}$ . The code  $\mathcal{C}$  is said to be *linear* if  $\mathcal{C}$  is an  $\Gamma$ -submodule of  $\Gamma^N$ . All codes in this paper are assumed to be linear. The ambient space  $\Gamma^N$  is equipped with the usual Euclidian inner product, i.e.,  $[a, b]_E = \sum_{j=0}^{N-1} a_j b_j$ , where  $a = (a_0, a_1, \dots, a_{N-1}), b = (b_0, b_1, \dots, b_{N-1}) \in \Gamma^N$ , and the *dual code* is defined by  $\mathcal{C}^{\perp_E} = \{a \in \Gamma^N \mid [a, b]_E = 0, \forall b \in \mathcal{C}\}$ . If  $\mathcal{C}^{\perp_E} = \mathcal{C}$ ,  $\mathcal{C}$  is called a *self-dual code* over  $\Gamma$ .

Let  $\gamma \in \Gamma^\times$  and  $\mathcal{C}$  be a linear code over  $\Gamma$  of length  $N$ .  $\mathcal{C}$  is called a  $\gamma$ -*constacyclic code* if  $(\gamma c_{N-1}, c_0, c_1, \dots, c_{N-2}) \in \mathcal{C}$  for all  $(c_0, c_1, \dots, c_{N-1}) \in \mathcal{C}$ . Particularly,  $\mathcal{C}$  is called a *negacyclic code* if  $\gamma = -1$ , and  $\mathcal{C}$  is called a *cyclic code* if  $\gamma = 1$ . For any  $a = (a_0, a_1, \dots, a_{N-1}) \in \Gamma^N$ , let  $a(x) = \sum_{i=0}^{N-1} a_i x^i \in \Gamma[x]/\langle x^N - \gamma \rangle$ . We will identify  $a$  with  $a(x)$  in this paper. By [5] Propositions 2.2 and 2.4, we have

**Lemma 1.1** *Let  $\gamma \in \Gamma^\times$ . Then  $\mathcal{C}$  is a  $\gamma$ -constacyclic code of length  $N$  over  $\Gamma$  if and only if  $\mathcal{C}$  is an ideal of the residue class ring  $\Gamma[x]/\langle x^N - \gamma \rangle$ .*

**Lemma 1.2** *The dual code of a  $\gamma$ -constacyclic code of length  $N$  over  $\Gamma$  is a  $\gamma^{-1}$ -constacyclic code of length  $N$  over  $\Gamma$ , i.e., an ideal of  $\Gamma[x]/\langle x^N - \gamma^{-1} \rangle$ .*

In this paper, let  $\mathbb{F}_q$  be a finite field of cardinality  $q$ , where  $q$  is power of a prime, and denote  $R = \mathbb{F}_q[u]/\langle u^e \rangle = \mathbb{F}_q + u\mathbb{F}_q + \dots + u^{e-1}\mathbb{F}_q$  ( $u^e = 0$ ) where  $e \geq 2$ . Let  $e = k \geq 3$ . For the case of  $p = 2$  and  $m = 1$  Abualrub and Siap [1] studied cyclic codes over the ring  $\mathbb{Z}_2 + u\mathbb{Z}_2$  and  $\mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2$  for arbitrary length  $N$ , then Al-Ashker and Hamoudeh [2] extended some of the results in [1], and studied cyclic codes of an arbitrary length over the ring  $\mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2 + \dots + u^{k-1}\mathbb{Z}_2$  ( $u^k = 0$ ). For the case of  $m = 1$ , Han et al. [7] studied cyclic codes over  $R = F_p + uF_p + \dots + u^{k-1}F_p$  with length  $p^s n$  using discrete Fourier transform. Singh et al. [9] studied cyclic code over the ring  $\mathbb{Z}_p[u]/\langle u^k \rangle = \mathbb{Z}_p + u\mathbb{Z}_p + u^2\mathbb{Z}_p + \dots + u^{k-1}\mathbb{Z}_p$  for any prime integer  $p$  and positive integer  $N$ . Kai et al. [8] investigated  $(1 + \lambda u)$ -constacyclic codes of arbitrary length over  $\mathbb{F}_p[u]/\langle u^m \rangle$ , where  $\lambda$  is a unit in  $\mathbb{F}_p[u]/\langle u^m \rangle$ , and Cao [3] generalized these results to  $(1 + w\gamma)$ -constacyclic codes of arbitrary length over an arbitrary finite chain ring  $R$ , where  $w$  is a unit of  $R$  and  $\gamma$  generates the unique maximal ideal of  $R$ . Sobhani et al. [10] showed that the Gray image of a  $(1 - u^{e-1})$ -constacyclic code of length  $n$  is a length  $p^{m(e-1)}n$  quasi-cyclic code of index  $p^{m(e-1)-1}$ .

Recently, Sobhani [11] determined the structure of  $(\delta + \alpha u^2)$ -constacyclic codes of length  $p^k$  over  $\mathbb{F}_{p^m}[u]/\langle u^3 \rangle$ , characterized and enumerated self-dual

codes among these codes, where  $\delta, \alpha \in \mathbb{F}_{p^m}^\times$ . Moreover, Sobhani proposed some open problems and further researches in this area:  
characterize  $(\delta + \alpha u^2)$ -constacyclic codes of length  $p^k$  over the finite chain ring  $\mathbb{F}_{p^m}[u]/\langle u^e \rangle$  for  $e \geq 4$ .

In this paper, we provide a new way different from the methods used in [7], [9] and [11] to determine the algebraic structures of a class of  $(\delta + \alpha u^2)$ -constacyclic codes over the finite chain ring  $\mathbb{F}_{p^m}[u]/\langle u^e \rangle$  for  $e = 4$ .

**Notation 1.3** Let  $\delta, \alpha \in \mathbb{F}_q^\times$  and  $n$  be a positive integer satisfying  $\gcd(q, n) = 1$ . We denote

- $R = \mathbb{F}_q[u]/\langle u^4 \rangle = \mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q + u^3\mathbb{F}_q$  ( $u^4 = 0$ ).
- $\mathcal{A} = \mathbb{F}_q[x]/\langle (x^n - \delta)^2 \rangle$ .
- $\mathcal{A}[v]/\langle v^2 - \alpha^{-1}(x^n - \delta) \rangle = \mathcal{A} + v\mathcal{A}$  ( $v^2 = \alpha^{-1}(x^n - \delta)$ ).

The present paper is organized as follows. In Section 2, we provide an explicit representation for each  $(\delta + \alpha u^2)$ -constacyclic code over  $R$  of length  $n$  and obtain a formula to count the number of codewords in each code from its representation. Then we give the dual code for each of such codes in Section 3. In Section 4, we determine all self-dual  $(1 + \alpha u^2)$ -constacyclic code over  $R$  of odd length  $n$  for the case of  $q = 2^m$ . Finally, we list all 125 distinct  $(1 + u^2)$ -constacyclic codes over  $\mathbb{F}_2[u]/\langle u^4 \rangle$  of length 7 in Section 5.

## 2. Representation for $(\delta + \alpha u^2)$ -constacyclic codes over $R$ of length $n$

In this section, we will construct a specific ring isomorphism from  $\mathcal{A} + v\mathcal{A}$  onto  $R[x]/\langle x^n - (\delta + \alpha u^2) \rangle$ . Hence we obtain a one-to-one correspondence between the set of ideals of  $\mathcal{A} + v\mathcal{A}$  onto the set of ideals of  $R[x]/\langle x^n - (\delta + \alpha u^2) \rangle$ , i.e., the set of  $(\delta + \alpha u^2)$ -constacyclic codes over  $R$  of length  $n$ .

By Notation 1.3,  $\mathcal{A} + v\mathcal{A} = \{\xi_0 + v\xi_1 \mid \xi_0, \xi_1 \in \mathcal{A}\}$  and the addition and multiplication are defined by

$$(\xi_0 + v\xi_1) + (\eta_0 + v\eta_1) = (\xi_0 + \eta_0) + v(\xi_1 + \eta_1),$$

$$(\xi_0 + v\xi_1)(\eta_0 + v\eta_1) = (\xi_0\eta_0 + \alpha^{-1}(x^n - \delta)\xi_1\eta_1) + v(\xi_0\eta_1 + \xi_1\eta_0),$$

for all  $\xi_0, \xi_1, \eta_0, \eta_1 \in \mathcal{A}$ .

Let  $\xi_0 + v\xi_1 \in \mathcal{A} + v\mathcal{A}$  where  $\xi_0, \xi_1 \in \mathcal{A}$ . It is clear that  $\xi_0$  can be uniquely expressed as  $\xi_0 = \xi_0(x)$  where  $\xi_0(x) \in \mathbb{F}_q[x]$  satisfying  $\deg(\xi_0(x)) < 2n$  (we

will write  $\deg(0) = -\infty$  for convenience). Dividing  $\xi_0(x)$  by  $\alpha^{-1}(x^n - \delta)$ , we obtain a unique pair  $(a_0(x), a_2(x))$  of polynomials in  $\mathbb{F}_q[x]$  such that

$$\xi_0 = \xi_0(x) = a_0(x) + \alpha^{-1}(x^n - \delta)a_2(x), \quad \deg(a_j(x)) < n$$

for  $j = 0, 2$ . Similarly, there is a unique pair  $(a_1(x), a_3(x))$  of polynomials in  $\mathbb{F}_q[x]$  such that

$$\xi_1 = \xi_1(x) = a_1(x) + \alpha^{-1}(x^n - \delta)a_3(x), \quad \deg(a_j(x)) < n$$

for  $j = 1, 3$ . Denote  $a_k(x) = \sum_{i=0}^{n-1} a_{i,k} x^i$  where  $a_{i,k} \in \mathbb{F}_q$  for all  $i = 0, 1, \dots, n-1$  and  $k = 0, 1, 2, 3$ . Then  $\xi_0 + v\xi_1$  can be uniquely written as a product of matrices:

$$\xi_0 + v\xi_1 = (1, x, \dots, x^{n-1})M \begin{pmatrix} 1 \\ v \\ \alpha^{-1}(x^n - \delta) \\ v\alpha^{-1}(x^n - \delta) \end{pmatrix},$$

where  $M = (a_{i,k})_{0 \leq i \leq n-1, 0 \leq k \leq 3}$  is an  $n \times 4$  matrix over  $\mathbb{F}_q$ . Now, we define

$$\Psi(\xi_0 + v\xi_1) = (1, x, \dots, x^{n-1})M \begin{pmatrix} 1 \\ u \\ u^2 \\ u^3 \end{pmatrix} = \sum_{i=0}^{n-1} \beta_i x^i,$$

where  $\beta_i = \sum_{k=0}^3 u^k a_{i,k} \in R$  for all  $i = 0, 1, \dots, n-1$ . Then it is clear that  $\Psi$  is a bijection from  $\mathcal{A} + v\mathcal{A}$  onto  $R[x]/\langle x^n - (\delta + \alpha u^2) \rangle$ . Furthermore, by  $v^2 = \alpha^{-1}(x^n - \delta)$ ,  $(x^n - \delta)^2 = 0$  in  $\mathcal{A} + v\mathcal{A}$  and  $x^n - (\delta + \alpha u^2) = 0$  in  $R[x]/\langle x^n - (\delta + \alpha u^2) \rangle$  one can easily verify the following conclusions.

**Theorem 2.1** *Using the notations above,  $\Psi$  is a ring isomorphism from  $\mathcal{A} + v\mathcal{A}$  onto  $R[x]/\langle x^n - (\delta + \alpha u^2) \rangle$ .*

**Remark** It is clear that both  $\mathcal{A} + v\mathcal{A}$  and  $R[x]/\langle x^n - (\delta + \alpha u^2) \rangle$  are  $\mathbb{F}_q$ -algebras of dimension  $4n$ . Specifically,  $\{1, x, \dots, x^{2n-1}, v, vx, \dots, vx^{2n-1}\}$  is an  $\mathbb{F}_q$ -basis of  $\mathcal{A} + v\mathcal{A}$ ,  $\cup_{k=0}^3 \{u^k, u^k x, \dots, u^k x^{n-1}\}$  is an  $\mathbb{F}_q$ -basis of  $R[x]/\langle x^n - (\delta + \alpha u^2) \rangle$  and  $\Psi$  is an  $\mathbb{F}_q$ -algebra isomorphism from  $\mathcal{A} + v\mathcal{A}$  onto  $R[x]/\langle x^n - (\delta + \alpha u^2) \rangle$  determined by:

$$\Psi(x^i) = x^i \text{ if } 0 \leq i \leq n-1, \quad \Psi(x^n) = \delta + \alpha u^2 \text{ and } \Psi(v) = u.$$

By Theorem 2.1, in order to determine all distinct  $(\delta + \alpha u^2)$ -constacyclic codes over  $R$  of length  $n$  it is sufficient to list all distinct ideals of  $\mathcal{A} + v\mathcal{A}$ . First, we study the structures of  $\mathcal{A}$  and  $\mathcal{A} + v\mathcal{A}$  in the following.

Since  $\delta \in \mathbb{F}_q^\times$  and  $\gcd(q, n) = 1$ , there are pairwise coprime monic irreducible polynomials  $f_1(x), \dots, f_r(x)$  in  $\mathbb{F}_q[x]$  such that

$$x^n - \delta = f_1(x) \dots f_r(x) \quad (1)$$

and  $(x^n - \delta)^2 = (x^n - \delta_0)^2 = f_1(x)^2 \dots f_r(x)^2$ . For any integer  $j$ ,  $1 \leq j \leq r$ , we assume  $\deg(f_j(x)) = d_j$  and denote  $F_j(x) = \frac{x^n - \delta}{f_j(x)}$ . Then  $F_j(x)^2 = \frac{(x^n - \delta)^2}{f_j(x)^2}$  and  $\gcd(F_j(x)^2, f_j(x)^2) = 1$ . Hence there exist  $g_j(x), h_j(x) \in \mathbb{F}_q[x]$  such that

$$g_j(x)F_j(x)^2 + h_j(x)f_j(x)^2 = 1. \quad (2)$$

From now on, we adopt the following notations.

**Notation 2.2** For any  $1 \leq j \leq r$ , let  $\varepsilon_j(x) \in \mathcal{A}$  be defined by

$$\varepsilon_j(x) \equiv g_j(x)F_j(x)^2 = 1 - h_j(x)f_j(x)^2 \pmod{(x^n - \delta)^2}$$

and denote  $\mathcal{K}_j = \mathbb{F}_q[x]/\langle f_j(x)^2 \rangle$ .

By the Chinese remainder theorem for commutative rings, we give the structure and properties of the ring  $\mathcal{A}$ .

**Lemma 2.3** *Using the notations above, we have the following:*

- (i)  $\varepsilon_1(x) + \dots + \varepsilon_r(x) = 1$ ,  $\varepsilon_j(x)^2 = \varepsilon_j(x)$  and  $\varepsilon_j(x)\varepsilon_l(x) = 0$  in the ring  $\mathcal{A}$  for all  $1 \leq j \neq l \leq r$ .
- (ii)  $\mathcal{A} = \mathcal{A}_1 \oplus \dots \oplus \mathcal{A}_r$  where  $\mathcal{A}_j = \mathcal{A}\varepsilon_j(x)$  with  $\varepsilon_j(x)$  as its multiplicative identity and satisfies  $\mathcal{A}_j\mathcal{A}_l = \{0\}$  for all  $1 \leq j \neq l \leq r$ .
- (iii) For any integer  $j$ ,  $1 \leq j \leq r$ , and  $a(x) \in \mathcal{K}_j$  we define  $\varphi_j : a(x) \mapsto \varepsilon_j(x)a(x) \pmod{(x^n - \delta)^2}$ . Then  $\varphi_j$  is a ring isomorphism from  $\mathcal{K}_j$  onto  $\mathcal{A}_j$ .
- (iv) For any  $a_j(x) \in \mathcal{K}_j$  for  $j = 1, \dots, r$ , define

$$\varphi(a_1(x), \dots, a_r(x)) = \sum_{j=1}^r \varphi_j(a_j(x)) = \sum_{j=1}^r \varepsilon_j(x)a_j(x)$$

$\pmod{(x^n - \delta)^2}$ . Then  $\varphi$  is a ring isomorphism from  $\mathcal{K}_1 \times \dots \times \mathcal{K}_r$  onto  $\mathcal{A}$ .

In order to describe the structure of  $\mathcal{A} + v\mathcal{A}$  ( $v^2 = \alpha^{-1}(x^n - \delta)$ ), we need the following lemma.

**Lemma 2.4** *Let  $1 \leq j \leq r$  and denote  $\omega_j = \alpha^{-1}F_j(x) \pmod{f_j(x)^2}$ . Then  $\omega_j$  is an invertible element of  $\mathcal{K}_j$  and satisfies  $\alpha^{-1}(x^n - \delta) = \omega_j f_j(x)$  in  $\mathcal{K}_j$ .*

**Proof.** Since  $\omega_j \in \mathcal{K}_j$  satisfying  $\omega_j \equiv \alpha^{-1}F_j(x) \pmod{f_j(x)^2}$ , by Equation (2) it follows that

$$\begin{aligned} (\alpha g_j(x)F_j(x))\omega_j &\equiv (\alpha g_j(x)F_j(x))(\alpha^{-1}F_j(x)) \\ &= 1 - h_j(x)f_j(x)^2 \\ &\equiv 1 \pmod{f_j(x)^2}, \end{aligned}$$

which implies that  $(\alpha g_j(x)F_j(x))\omega_j = 1$  in the ring  $\mathcal{K}_j$ . Hence  $\omega_j \in \mathcal{K}_j^\times$  and  $\omega_j^{-1} = \alpha g_j(x)F_j(x) \pmod{f_j(x)^2}$ . By Equation (1) and  $F_j(x) = \frac{x^n - \delta}{f_j(x)}$ , we deduce that  $\alpha^{-1}(x^n - \delta) = \alpha^{-1}f_1(x) \dots f_r(x) = \alpha^{-1}F_j(x)f_j(x) = \omega_j f_j(x)$ .  $\square$

Now, we can provide the structure of  $\mathcal{A} + v\mathcal{A}$ .

**Lemma 2.5** *Let  $1 \leq j \leq r$ . Using the notations in Lemma 2.4, we denote*

$$\begin{aligned} \mathcal{K}_j[v]/\langle v^2 - \omega_j f_j(x) \rangle &= \mathcal{K}_j + v\mathcal{K}_j \quad (v^2 = \omega_j f_j(x)), \\ \mathcal{A}_j + v\mathcal{A}_j &= \varepsilon_j(x)(\mathcal{A} + v\mathcal{A}) \quad (v^2 = \alpha^{-1}(x^n - \delta)). \end{aligned}$$

*Then we have the following conclusions:*

(i)  $\mathcal{A} + v\mathcal{A} = (\mathcal{A}_1 + v\mathcal{A}_1) \oplus \dots \oplus (\mathcal{A}_r + v\mathcal{A}_r)$ , where  $\varepsilon_j(x)$  is the multiplicative identity of the ring  $\mathcal{A}_j + v\mathcal{A}_j$  and this decomposition satisfies  $(\mathcal{A}_j + v\mathcal{A}_j)(\mathcal{A}_l + v\mathcal{A}_l) = \{0\}$  for all  $1 \leq j \neq l \leq r$ .

(ii) For any  $1 \leq j \leq r$  and  $a(x), b(x) \in \mathcal{K}_j$ , we define

$$\Phi_j : a(x) + vb(x) \mapsto \varepsilon_j(x)(a(x) + vb(x)) \pmod{(x^n - \delta)^2}.$$

*Then  $\Phi_j$  is a ring isomorphism from  $\mathcal{K}_j + v\mathcal{K}_j$  onto  $\mathcal{A}_j + v\mathcal{A}_j$ .*

(iii) For any  $\beta_j, \gamma_j \in \mathcal{K}_j$ ,  $j = 1, \dots, r$ , define

$$\Phi(\beta_1 + v\gamma_1, \dots, \beta_r + v\gamma_r) = \sum_{j=1}^r \Phi_j(\beta_j + v\gamma_j) = \sum_{j=1}^r \varepsilon_j(x)(\beta_j + v\gamma_j).$$

*Then  $\Phi$  is a ring isomorphism from  $(\mathcal{K}_1 + v\mathcal{K}_1) \times \dots \times (\mathcal{K}_r + v\mathcal{K}_r)$  onto  $\mathcal{A} + v\mathcal{A}$ .*

**Proof.** (i) Since  $\varepsilon_j(x)$  is an element of  $\mathcal{A}$  and  $\mathcal{A}$  is a subring of  $\mathcal{A} + v\mathcal{A}$ ,  $\varepsilon_j(x)$  is also an idempotent of the ring  $\mathcal{A} + v\mathcal{A}$  for all  $j = 1, \dots, r$ . Then the conclusions follow from Lemma 2.3(i) and classical ring theory.

(ii) For any  $a(x), b(x) \in \mathcal{K}_j$ , by the definition of  $\varphi_j$  in Lemma 2.3(iii) it follows that

$$\begin{aligned}\Phi_j(a(x) + vb(x)) &= (\varepsilon_j(x)a(x)) + v(\varepsilon_j(x)b(x)) \\ &= \varphi_j(a(x)) + v\varphi_j(b(x)).\end{aligned}$$

Hence  $\Phi_j$  is a bijection from  $\mathcal{K}_j + v\mathcal{K}_j$  onto  $\mathcal{A}_j + v\mathcal{A}_j$  by Lemma 2.3(iii).

Let  $a_1(x), b_1(x), a_2(x), b_2(x) \in \mathcal{K}_j$ , and denote  $\xi_i = a_i(x) + vb_i(x) \in \mathcal{K}_j + v\mathcal{K}_j$  for  $i = 1, 2$ . Since  $\varphi_j$  is a ring isomorphism from  $\mathcal{K}_j$  onto  $\mathcal{A}_j$ , by Lemmas 2.4 and 2.5 we have that  $\Phi_j(\xi_1 + \xi_2) = \Phi_j(\xi_1) + \Phi_j(\xi_2)$  and

$$\begin{aligned}\Phi_j(\xi_1\xi_2) &= \Phi_j((a_1(x)b_1(x) + \omega_j f_j(x)a_2(x)b_2(x)) \\ &\quad + v(a_1(x)b_2(x) + a_2(x)b_1(x))) \\ &= (\varepsilon_j(x)a_1(x))(\varepsilon_j(x)b_1(x)) \\ &\quad + \alpha^{-1}(x^n - \delta)(\varepsilon_j(x)a_2(x))(\varepsilon_j(x)b_2(x)) \\ &\quad + v((\varepsilon_j(x)a_1(x))(\varepsilon_j(x)b_2(x)) \\ &\quad + (\varepsilon_j(x)a_2(x))(\varepsilon_j(x)b_1(x))) \\ &= \Phi_j(\xi_1)\Phi_j(\xi_2).\end{aligned}$$

Therefore,  $\Phi_j$  is a ring isomorphism from  $\mathcal{K}_j + v\mathcal{K}_j$  onto  $\mathcal{A}_j + v\mathcal{A}_j$ .

(iii) It follows from (i) and (ii) immediately.  $\square$

In order to determine all ideals of  $\mathcal{A} + v\mathcal{A}$ , by Lemma 2.5(iii) and classical ring theory it is sufficient to list all distinct ideals of  $\mathcal{K}_j + v\mathcal{K}_j$  ( $v^2 = \omega_j f_j(x)$ ) for all  $j = 1, \dots, r$ . To do this, we need the following lemma.

**Lemma 2.6** (cf. [4] Example 2.1) *Let  $1 \leq j \leq r$ . Then we have the following:*

(i)  $\mathcal{K}_j$  is a finite chain ring,  $f_j(x)$  generates the unique maximal ideal  $\langle f_j(x) \rangle = f_j(x)\mathcal{K}_j$  of  $\mathcal{K}_j$ , the nilpotency index of  $f_j(x)$  is equal to 2 and the residue class field of  $\mathcal{K}_j$  modulo  $\langle f_j(x) \rangle$  is  $\mathcal{K}_j/\langle f_j(x) \rangle \cong \mathbb{F}_q[x]/\langle f_j(x) \rangle$ , where  $\mathbb{F}_q[x]/\langle f_j(x) \rangle$  is an extension field of  $\mathbb{F}_q$  with  $q^{d_j}$  elements.

(ii) Let  $\mathcal{T}_j = \{\sum_{i=0}^{d_j-1} t_i x^i \mid t_0, t_1, \dots, t_{d_j-1} \in \mathbb{F}_q\}$ . Then  $|\mathcal{T}_j| = q^{d_j}$  and every element  $\xi$  of  $\mathcal{K}_j$  has a unique  $f_j(x)$ -adic expansion:  $\xi = b_0(x) + f_j(x)b_1(x)$ ,  $b_0(x), b_1(x) \in \mathcal{T}_j$ . Hence  $|\mathcal{K}_j| = |\mathcal{T}_j|^2 = q^{2d_j}$ . Moreover,  $\xi \in \mathcal{K}_j^\times$  if and only if  $b_0(x) \neq 0$ .

Then we determine ideals of  $\mathcal{K}_j + v\mathcal{K}_j$  ( $v^2 = \omega_j f_j(x)$ ).

**Lemma 2.7** *Let  $1 \leq j \leq r$ . Then all distinct ideals of  $\mathcal{K}_j + v\mathcal{K}_j$  are given by:  $\langle v^l \rangle$ ,  $l = 0, 1, 2, 3, 4$ . Moreover, the number of elements contained in  $\langle v^l \rangle$  is equal to  $|\langle v^l \rangle| = q^{(4-l)d_j}$ .*

**Proof.** Let  $\xi_0 + v\xi_1 \in \mathcal{K}_j + v\mathcal{K}_j$  where  $\xi_0, \xi_1 \in \mathcal{K}_j$ . By Lemma 2.6(ii), each  $\xi_i$  has a unique  $f_j(x)$ -expansion:  $\xi_i = b_{i,0}(x) + f_j(x)b_{i,1}(x)$ ,  $b_{i,0}(x), b_{i,1}(x) \in \mathcal{T}_j$ , where  $i = 0, 1$ . By  $f_j(x) = v^2\omega_j^{-1}$  in the ring  $\mathcal{K}_j + v\mathcal{K}_j$ , it follows that

$$\begin{aligned} \xi_0 + v\xi_1 &= b_{0,0}(x) + v^2\omega_j^{-1}b_{0,1}(x) \\ &\quad + v(b_{1,0}(x) + v^2\omega_j^{-1}b_{1,1}(x)). \end{aligned} \quad (3)$$

By the proof of Lemma 2.4, we see that  $\omega_j^{-1} = \alpha g_j(x)F_j(x) \pmod{f_j(x)^2}$ . Dividing  $\omega_j^{-1}b_{i,1}(x)$  by  $f_j(x)$  we obtain a unique polynomial  $h_i(x) \in \mathcal{T}_j$  such that

$$\omega_j^{-1}b_{i,1}(x) = f_j(x)a_i(x) + h_i(x)$$

for some  $a_i(x) \in \mathbb{F}_q[x]$ . From this and by  $f_j(x)^2 = 0$  in  $\mathcal{K}_j$ , we deduce that

$$v^2\omega_j^{-1}b_{i,1}(x) = v^2h_i(x) + f_j(x) \cdot f_j(x)a_i(x) = v^2h_i(x), \quad i = 0, 1.$$

Then by (3) we obtain the following  $v$ -expansion for  $\xi_0 + v\xi_1$ :

$$\xi_0 + v\xi_1 = b_{0,0}(x) + vb_{1,0}(x) + v^2h_0(x) + v^3h_1(x).$$

Obviously,  $v^4 = (v^2)^2 = \omega_j^2 f_j(x)^2 = 0$  and  $v^3 = v\omega_j(x)f_j(x) \neq 0$ . Hence the nilpotency index of  $v$  is equal to 4 in  $\mathcal{K}_j + v\mathcal{K}_j$ . Moreover, by Lemma 2.6(ii) we see that  $\xi_0 + v\xi_1$  is invertible if and only if  $b_{0,0}(x) \neq 0$ .

As stated above, we conclude that  $v$  generates the unique maximal ideal  $\langle v \rangle$  of  $\mathcal{K}_j + v\mathcal{K}_j$  and the residue class field is  $(\mathcal{K}_j + v\mathcal{K}_j)/\langle v \rangle = \{b_{0,0}(x) + \langle v \rangle \mid b_{0,0}(x) \in \mathcal{T}_j\}$  satisfying  $|(\mathcal{K}_j + v\mathcal{K}_j)/\langle v \rangle| = |\mathcal{T}_j| = q^{d_j}$ . Therefore, all distinct ideals of  $\mathcal{K}_j + v\mathcal{K}_j$  are given by:

$$\{0\} = \langle v^4 \rangle \subset \langle v^3 \rangle \subset \langle v^2 \rangle \subset \langle v \rangle \subset \langle v^0 \rangle = \mathcal{K}_j + v\mathcal{K}_j.$$

Furthermore, for any  $0 \leq l \leq 4$  we have that

$$\langle v^l \rangle = \left\{ \sum_{k=l}^4 v^k t_k(x) \mid t_k(x) \in \mathcal{T}_j, \quad k = l, \dots, 3 \right\},$$



which implies  $|\langle v^l \rangle| = |\mathcal{T}_j|^{4-l} = q^{(4-l)d_j}$ .  $\square$

Since  $\Psi$  is a ring isomorphism from  $\mathcal{A} + v\mathcal{A}$  onto  $R[x]/\langle x^n - (\delta + \alpha u^2) \rangle$ , by Lemma 2.3(i) and the definition of  $\Psi$  we deduce the following corollary.

**Corollary 2.8** *For any integer  $j$ ,  $1 \leq j \leq r$ , denote  $e_j(x) = \Psi(\varepsilon_j(x)) \in R[x]/\langle x^n - (\delta + \alpha u^2) \rangle$ . Then*

- (i)  $e_1(x) + \dots + e_r(x) = 1$ ,  $e_j(x)^2 = e_j(x)$  and  $e_j(x)e_l(x) = 0$  in the ring  $R[x]/\langle x^n - (\delta + \alpha u^2) \rangle$  for all  $1 \leq j \neq l \leq r$ .
- (ii) If  $\varepsilon_j(x) = e_{j,0}(x) + \alpha^{-1}(x^n - \delta)e_{j,1}(x)$  where  $e_{j,i}(x) \in \mathbb{F}_q[x]$  satisfying  $\deg(e_{j,i}(x)) \leq n-1$  for  $i = 0, 1$ , then  $e_j(x) = e_{j,0}(x) + u^2 e_{j,1}(x)$ .

Finally, we give a precise representation for any  $(\delta + \alpha u^2)$ -constacyclic code over  $R$  of length  $n$ .

**Theorem 2.9** *Using the notations above, all distinct  $(\delta + \alpha u^2)$ -constacyclic codes over  $R$  of length  $n$  are given by:*

$$\mathcal{C}_{(l_1, \dots, l_r)} = \left\langle \sum_{j=1}^r u^{l_j} e_j(x) \right\rangle, \quad 0 \leq l_1, \dots, l_r \leq 4.$$

Moreover, the number of codewords contained in  $\mathcal{C}_{(l_1, \dots, l_r)}$  is equal to  $|\mathcal{C}_{(l_1, \dots, l_r)}| = q^{\sum_{j=1}^r (4-l_j)d_j}$ .

Therefore, the number of  $(\delta + \alpha u^2)$ -constacyclic codes over  $R$  of length  $n$  is equal to  $5^r$ .

**Proof.** By Theorem 2.1 and Lemma 2.5(iii), we see that  $\Psi \circ \Phi$  is a ring isomorphism from  $(\mathcal{K}_1 + v\mathcal{K}_1) \times \dots \times (\mathcal{K}_r + v\mathcal{K}_r)$  onto  $R[x]/\langle x^n - (\delta + \alpha u^2) \rangle$ . Let  $\mathcal{C}$  be an ideal of  $R[x]/\langle x^n - (\delta + \alpha u^2) \rangle$ . By classical ring theory and Lemma 2.7, for any integer  $j$ ,  $1 \leq j \leq r$ , there is a unique ideal  $\langle v^{l_j} \rangle$  of  $\mathcal{K}_j + v\mathcal{K}_j$ , where  $0 \leq l_j \leq 4$ , such that

$$\begin{aligned} \mathcal{C} &= (\Psi \circ \Phi) (\langle v^{l_1} \rangle \times \dots \times \langle v^{l_r} \rangle) \\ &= \Psi (\Phi \{ (\xi_1, \dots, \xi_r) \mid \xi_j \in \langle v^{l_j} \rangle, j = 1, \dots, r \}) \\ &= \Psi \left( \left\{ \sum_{j=1}^r \varepsilon_j(x) \xi_j \mid \xi_j \in \langle v^{l_j} \rangle, j = 1, \dots, r \right\} \right) \\ &= \Psi \left( \bigoplus_{j=1}^r \langle \varepsilon_j(x) v^{l_j} \rangle \right) = \bigoplus_{j=1}^r \langle \Psi(\varepsilon_j(x) v^{l_j}) \rangle. \end{aligned}$$

Since  $\Psi \circ \Phi$  is a ring isomorphism from  $(\mathcal{K}_1 + v\mathcal{K}_1) \times \dots \times (\mathcal{K}_r + v\mathcal{K}_r)$  onto  $R[x]/\langle x^n - (\delta + \alpha u^2) \rangle$ , by Lemma 2.7 we have  $|\mathcal{C}| = |\langle v^{l_1} \rangle \times \dots \times \langle v^{l_r} \rangle| = \prod_{j=1}^r |\langle v^{l_j} \rangle| = q^{\sum_{j=1}^r (4-l_j)d_j}$ .

By Corollary 2.8 and the remark after Theorem 2.1, we deduce that  $\Psi(\varepsilon_j(x)v^{l_j}) = \Psi(\varepsilon_j(x))\Psi(v^{l_j}) = u^{l_j}e_j(x)$  for all  $j = 1, \dots, r$ , which implies

$$\mathcal{C} = \bigoplus_{j=1}^r \langle u^{l_j}e_j(x) \rangle = \langle u^{l_1}e_1(x), \dots, u^{l_r}e_r(x) \rangle.$$

From this and by Corollary 2.8(i), one can easily verify that  $\mathcal{C} = \langle u^{l_1}e_1(x) + \dots + u^{l_r}e_r(x) \rangle$ .

As stated above, we conclude that the number of  $(\delta + \alpha u^2)$ -constacyclic codes over  $R$  of length  $n$  is equal to  $5^r$  by Lemma 2.7.  $\square$

### 3. Dual codes of $(\delta + \alpha u^2)$ -constacyclic codes $\mathcal{C}$ over $R$ of length $n$

In this section, we give the dual code  $\mathcal{C}^{\perp_E}$  of any  $(\delta + \alpha u^2)$ -constacyclic code  $\mathcal{C}$  over  $R = \mathbb{F}_q[u]/\langle u^4 \rangle$  of length  $n$ , where  $\delta, \alpha \in \mathbb{F}_q^\times$  and  $\gcd(q, n) = 1$ . By Lemma 1.2, we know that  $\mathcal{C}^{\perp_E}$  is a  $(\delta + \alpha u^2)^{-1}$ -constacyclic code over  $R$  of length  $n$ , i.e.,  $\mathcal{C}^{\perp_E}$  is an ideal of the ring  $R[x]/\langle x^n - (\delta + \alpha u^2)^{-1} \rangle$ .

In the ring  $R[x]/\langle x^n - (\delta + \alpha u^2)^{-1} \rangle$ , we have  $x^n = (\delta + \alpha u^2)^{-1}$ , i.e.,  $x^{-n} = \delta + \alpha u^2$  or  $(\delta + \alpha u^2)x^n = 1$ , which implies

$$x^{-1} = (\delta + \alpha u^2)x^{n-1} \text{ in } R[x]/\langle x^n - (\delta + \alpha u^2)^{-1} \rangle. \quad (4)$$

**Lemma 3.1** *Define a map  $\tau : R[x]/\langle x^n - (\delta + \alpha u^2) \rangle \rightarrow R[x]/\langle x^n - (\delta + \alpha u^2)^{-1} \rangle$  by the rule that*

$$\tau(a(x)) = a(x^{-1}) = \sum_{i=0}^{n-1} a_i x^{-i} = (\delta + \alpha u^2) \sum_{i=0}^{n-1} a_i x^{n-i}, \quad (5)$$

for all  $a(x) = \sum_{i=0}^{n-1} a_i x^i \in R[x]/\langle x^n - (\delta + \alpha u^2) \rangle$  with  $a_0, a_1, \dots, a_{n-1} \in R$ . Then  $\tau$  is a ring isomorphism from  $R[x]/\langle x^n - (\delta + \alpha u^2) \rangle$  onto  $R[x]/\langle x^n - (\delta + \alpha u^2)^{-1} \rangle$ .

**Proof.** For any  $g(x) \in R[x]$ , we define

$$\tau_0(g(x)) = g((\delta + \alpha u^2)x^{n-1}) = g(x^{-1}) \pmod{x^n - (\delta + \alpha u^2)^{-1}}.$$

Then by Equation (4), we see that  $\tau_0$  is a well-defined ring homomorphism from  $R[x]$  to  $R[x]/\langle x^n - (\delta + \alpha u^2)^{-1} \rangle$ . For any  $h(x) = \sum_{i=0}^{n-1} h_i x^i \in R[x]/\langle x^n - (\delta + \alpha u^2)^{-1} \rangle$ , we select  $g(x) = (\delta + \alpha u^2)^{-1} \sum_{i=0}^{n-1} h_i x^{n-i} \in R[x]$ . Then by (4) and the definition of  $\tau_0$ , it follows that

$$\begin{aligned} \tau_0(g(x)) &= (\delta + \alpha u^2)^{-1} \sum_{i=0}^{n-1} h_i ((\delta + \alpha u^2) x^{n-1})^{n-i} \\ &= x^n \sum_{i=0}^{n-1} h_i x^{i-n} = h(x). \end{aligned}$$

Hence  $\tau_0$  is surjective. Then from

$$\tau_0(x^n - (\delta + \alpha u^2)) = x^{-n} - (\delta + \alpha u^2) = (\delta + \alpha u^2) - (\delta + \alpha u^2) = 0$$

in  $R[x]/\langle x^n - (\delta + \alpha u^2)^{-1} \rangle$  and by classical ring theory, we deduce that the map  $\tau$  induced by  $\tau_0$ , which is defined by (5), is a surjective ring homomorphism from  $R[x]/\langle x^n - (\delta + \alpha u^2) \rangle$  onto  $R[x]/\langle x^n - (\delta + \alpha u^2)^{-1} \rangle$ . Moreover, it is clear that  $|R[x]/\langle x^n - (\delta + \alpha u^2) \rangle| = |R|^n = |R[x]/\langle x^n - (\delta + \alpha u^2)^{-1} \rangle|$ . Therefore,  $\tau$  is a bijection and hence a ring isomorphism.  $\square$

**Lemma 3.2** For any  $a = (a_0, a_1, \dots, a_{n-1}) \in R^n$  and  $b = (b_0, b_1, \dots, b_{n-1}) \in R^n$ , denote

$$\begin{aligned} a(x) &= \sum_{i=0}^{n-1} a_i x^i \in R[x]/\langle x^n - (\delta + \alpha u^2) \rangle, \\ b(x) &= \sum_{i=0}^{n-1} b_i x^i \in R[x]/\langle x^n - (\delta + \alpha u^2)^{-1} \rangle. \end{aligned}$$

Then  $[a, b]_E = \sum_{i=0}^{n-1} a_i b_i = 0$  if  $\tau(a(x)) \cdot b(x) = 0$  in  $R[x]/\langle x^n - (\delta + \alpha u^2)^{-1} \rangle$ .

**Proof.** By Equation (5) and  $x^n = (\delta + \alpha u^2)^{-1}$  in  $R[x]/\langle x^n - (\delta + \alpha u^2)^{-1} \rangle$ , it follows that  $\tau(a(x)) \cdot b(x) = [a, b]_E + \sum_{i=1}^{n-1} c_i x^i$  for some  $c_1, \dots, c_{n-1} \in R$ . Hence  $[a, b]_E = 0$  if  $\tau(a(x)) \cdot b(x) = 0$  in  $R[x]/\langle x^n - (\delta + \alpha u^2)^{-1} \rangle$ .  $\square$

**Remark** For any  $(\delta + \alpha u^2)$ -constacyclic code  $\mathcal{C}$  over  $R$  of length  $n$ , by Lemma 3.2 it follows that  $\{b(x) \in R[x]/\langle x^n - (\delta + \alpha u^2)^{-1} \rangle \mid \tau(a(x)) \cdot b(x) = 0, \forall a(x) \in \mathcal{C}\} \subseteq \mathcal{C}^{\perp_E}$ .

Now, we determine the dual code of each  $(\delta + \alpha u^2)$ -constacyclic code over  $R$  of length  $n$ .

**Theorem 3.3** *Let  $\mathcal{C} = \langle \sum_{j=1}^r u^{l_j} e_j(x) \rangle$  be a  $(\delta + \alpha u^2)$ -constacyclic code over  $R$  of length  $n$  given by Theorem 2.9. Then the dual code of  $\mathcal{C}$  is given by:*

$$\mathcal{C}^{\perp_E} = \left\langle \sum_{j=1}^r u^{4-l_j} e_j(x^{-1}) \right\rangle,$$

which is an ideal of the ring  $R[x]/\langle x^n - (\delta + \alpha u^2)^{-1} \rangle$ .

**Proof.** Let  $\mathcal{D} = \langle \sum_{j=1}^r u^{4-l_j} e_j(x) \rangle$  be the ideal of  $R[x]/\langle x^n - (\delta + \alpha u^2) \rangle$  generated by  $\sum_{j=1}^r u^{4-l_j} e_j(x)$ . Since  $\tau$  is a ring isomorphism from  $R[x]/\langle x^n - (\delta + \alpha u^2) \rangle$  onto  $R[x]/\langle x^n - (\delta + \alpha u^2)^{-1} \rangle$ ,  $\tau(\mathcal{D})$  is an ideal of  $R[x]/\langle x^n - (\delta + \alpha u^2)^{-1} \rangle$ . From this and by Theorem 2.9, we deduce that

$$|\tau(\mathcal{D})| = |\mathcal{D}| = q^{\sum_{j=1}^r (4-(4-l_j))d_j} = q^{\sum_{j=1}^r l_j d_j}. \quad (6)$$

Moreover, by Corollary 2.8(i) it follows that

$$\begin{aligned} \tau(\mathcal{C}) \cdot \tau(\mathcal{D}) &= \tau(\mathcal{C} \cdot \mathcal{D}) \\ &= \tau \left( \left\langle \sum_{j=1}^r u^{l_j} e_j(x) \right\rangle \cdot \left\langle \sum_{j=1}^r u^{4-l_j} e_j(x) \right\rangle \right) \\ &= \tau \left( \left\langle \left( \sum_{j=1}^r u^{l_j} e_j(x) \right) \left( \sum_{j=1}^r u^{4-l_j} e_j(x) \right) \right\rangle \right) \\ &= \tau \left( \left\langle \sum_{j=1}^r (u^{l_j} u^{4-l_j}) e_j(x) \right\rangle \right) \\ &= \tau(\{0\}) = \{0\}. \end{aligned}$$

From this and by Lemma 3.2, we deduce that  $\tau(\mathcal{D}) \subseteq \mathcal{C}^{\perp_E}$ . Furthermore, by (6) and Theorem 2.9 it follows that

$$|\mathcal{C}| |\tau(\mathcal{D})| = q^{\sum_{j=1}^r (4-l_j)d_j} q^{\sum_{j=1}^r l_j d_j} = q^{4 \sum_{j=1}^r d_j} = q^{4n} = |R|^n.$$

Hence we conclude that  $\mathcal{C}^{\perp_E} = \tau(\mathcal{D})$  since  $R$  is a finite chain ring (cf. [6]). Finally, since  $\tau$  is a ring isomorphism defined in Lemma 3.1, we see that  $\mathcal{C}^{\perp_E} = \langle \tau(\sum_{j=1}^r u^{4-l_j} e_j(x)) \rangle = \langle \sum_{j=1}^r u^{4-l_j} \tau(e_j(x)) \rangle = \langle \sum_{j=1}^r u^{4-l_j} e_j(x^{-1}) \rangle$ .  $\square$

#### 4. Self-dual $(1 + \alpha u^2)$ -constacyclic codes over $\mathbb{F}_{2^m} + u\mathbb{F}_{2^m} + u^2\mathbb{F}_{2^m} + u^3\mathbb{F}_{2^m}$ of odd length

In this section, let  $q = 2^m$  where  $m$  is a positive integer,  $R = \mathbb{F}_{2^m}[u]/\langle u^4 \rangle$ ,  $n$  be an odd positive integer and  $\alpha \in \mathbb{F}_{2^m}^\times$ . As  $(1 + \alpha u^2)^{-1} = 1 + \alpha u^2$  in  $R$ , the dual code of every  $(1 + \alpha u^2)$ -constacyclic code over  $R$  of length  $n$  is also a  $(1 + \alpha u^2)$ -constacyclic code over  $R$  of length  $n$  by Lemma 1.2.

Moreover, by  $(1 + \alpha u^2)^{-1} = 1 + \alpha u^2$  in  $R$  and Lemma 3.1, we see that the map  $\tau$  defined by  $\tau(a(x)) = a(x^{-1})$  ( $\forall a(x) \in R[x]/\langle x^n - (1 + \alpha u^2) \rangle$ ) is a ring automorphism on  $R[x]/\langle x^n - (1 + \alpha u^2) \rangle$  satisfying  $\tau^{-1} = \tau$ . From this and by Corollary 2.8(i), we deduce that for each integer  $j$ ,  $1 \leq j \leq r$ , there is a unique integer  $j'$ ,  $1 \leq j' \leq r$ , such that

$$\tau(e_j(x)) = e_j(x^{-1}) = e_{j'}(x) \pmod{x^n - (1 + \alpha u^2)}.$$

Hence the ring automorphism  $\tau$  on  $R[x]/\langle x^n - (1 + \alpha u^2) \rangle$  induces a permutation  $j \mapsto j'$  on the set  $\{1, \dots, r\}$ . In order to simplify notations, we still denote this bijection by  $\tau$ , i.e.,

$$\tau(e_j(x)) = e_j(x^{-1}) = e_{\tau(j)}(x) \pmod{x^n - (1 + \alpha u^2)}. \quad (7)$$

Since the permutation  $\tau$  on  $\{1, \dots, r\}$  satisfies  $\tau^{-1} = \tau$ , After a suitable rearrangement of  $e_1(x), \dots, e_r(x)$ , there are nonnegative integers  $\rho, \epsilon$  such that  $\rho + 2\epsilon = r$  and

- $\tau(j) = j$ ,  $1 \leq j \leq \rho$ ;
- $\tau(\rho + i) = \rho + \epsilon + i$  and  $\tau(\rho + \epsilon + i) = \rho + i$ ,  $1 \leq i \leq \epsilon$ .

Now, by Theorem 3.3 and Equation (7), we obtain the following corollary immediately.

**Corollary 4.1** *Let  $\mathcal{C} = \langle \sum_{j=1}^r u^{l_j} e_j(x) \rangle$  be a  $(1 + \alpha u^2)$ -constacyclic code over  $R$  of length  $n$  given by Theorem 2.9. Then  $\mathcal{C}^{\perp_E} = \langle \sum_{j=1}^r u^{4-l_j} e_{\tau(j)}(x) \rangle$ .*

Finally, we list all self-dual  $(1 + \alpha u^2)$ -constacyclic codes over  $R$  of length  $n$  as follows.

**Theorem 4.2** *All distinct self-dual  $(1 + \alpha u^2)$ -constacyclic codes of length  $n$  over  $R$  are given by:*

$$\left\langle \sum_{j=1}^{\rho} u^2 e_j(x) + \sum_{i=1}^{\epsilon} (u^{l_{\rho+i}} e_{\rho+i}(x) + u^{4-l_{\rho+i}} e_{\rho+i+\epsilon}(x)) \right\rangle.$$

Therefore, the number of self-dual  $(1 + \alpha u^2)$ -constacyclic codes over  $R$  of length  $n$  is equal to  $5^\epsilon$ .

**Proof.** Let  $\mathcal{C}$  be any  $(1 + \alpha u^2)$ -constacyclic code over  $R$  of length  $n$ . By Theorem 2.9 and its proof, we have that  $\mathcal{C} = \bigoplus_{j=1}^r \langle u^{l_j} e_j(x) \rangle$  where  $0 \leq l_j \leq 4$  for all  $j = 1, \dots, r$ . By Corollary 4.1 and the proof of Theorem 2.9, we see that  $\mathcal{C}^{\perp_E} = \bigoplus_{j=1}^r \langle u^{4-l_j} e_{\tau(j)}(x) \rangle$ . Since  $\tau$  is a bijection on the set  $\{1, \dots, r\}$ , we have  $\mathcal{C} = \bigoplus_{j=1}^r \langle u^{l_{\tau(j)}} e_{\tau(j)}(x) \rangle$ . From this and by Corollary 2.8(i), we deduce that  $\mathcal{C}$  is self-dual if and only if  $l_{\tau(j)} = 4 - l_j$  for all  $j = 1, \dots, r$ . Then we have one of the following two cases.

(i) Let  $1 \leq j \leq \rho$ . Then  $\tau(j) = j$ , and  $l_{\tau(j)} = 4 - l_j$  if and only if  $l_j = 4 - l_j$ , which is equivalent to that  $l_j = 2$ .

(ii) Let  $j = \rho + i$ , where  $1 \leq i \leq \epsilon$ . In this case,  $\tau(j) = j + \epsilon$  and  $\tau(j + \epsilon) = j$ . Then  $l_{\tau(j)} = 4 - l_j$  and  $l_{\tau(j+\epsilon)} = 4 - l_{j+\epsilon}$ , i.e.,  $l_{j+\epsilon} = 4 - l_j$  and  $l_j = 4 - l_{j+\epsilon}$ , if and only if  $l_{j+\epsilon} = 4 - l_j$ .  $\square$

## 5. An example

Let  $\mathbb{F}_2 = \{0, 1\}$  and  $R = \mathbb{F}_2[u]/\langle u^4 \rangle = \mathbb{F}_2 + u\mathbb{F}_2 + u^2\mathbb{F}_2 + u^3\mathbb{F}_2$  ( $u^4 = 0$ ). Then  $R$  is a finite chain ring of  $2^4 = 16$  elements. It is known that  $x^7 - 1 = x^7 + 1 = f_1(x)f_2(x)f_3(x)$  where

$$f_1(x) = x + 1, f_2(x) = x^3 + x + 1 \text{ and } f_3(x) = x^3 + x^2 + 1$$

are irreducible polynomials in  $\mathbb{F}_2[x]$ . Obviously,  $d_j = \deg(f_j(x))$  where  $d_1 = 1$  and  $d_2 = d_3 = 3$ .

For  $1 \leq j \leq 3$ , let  $F_j(x) = \frac{x^7+1}{f_j(x)}$  and find  $g_j(x), h_j(x) \in \mathbb{F}_2[x]$  such that  $g_j(x)F_j(x)^2 + h_j(x)f_j(x)^2 = 1$ . Then we calculate  $\varepsilon_j(x) = g_j(x)F_j(x)^2 \pmod{x^{14}+1}$ . Dividing  $\varepsilon_j(x)$  by  $x^7 + 1$ , we obtain a unique pair  $(e_{j,0}(x), e_{j,1}(x))$  of polynomials in  $\mathbb{F}_2[x]$  such that  $\varepsilon_j(x) = e_{j,0}(x) + (x^7 + 1)e_{j,1}(x)$  and  $\deg(e_{j,i}(x)) \leq 6$  for  $i = 0, 1$ . Then we have

$$e_j(x) = e_{j,0}(x) + u^2 e_{j,1}(x) \in R[x]/\langle x^7 - (1 + u^2) \rangle$$

by Corollary 2.8(ii). Precisely, we have

$$e_1(x) = x^6 + (u^2 + 1)x^5 + x^4 + (u^2 + 1)x^3 + x^2 + (u^2 + 1)x + 1;$$

$$e_2(x) = x^4 + x^2 + (u^2 + 1)x + 1;$$

$$e_3(x) = x^6 + (u^2 + 1)x^5 + (u^2 + 1)x^3 + 1,$$

where  $\tau(e_1(x)) = e_1(x^{-1}) = e_1(x)$  and  $\tau(e_2(x)) = e_2(x^{-1}) = e_3(x) \pmod{x^7 - (1 + u^2)}$ . Hence  $\rho = \epsilon = 1$ .

- By Theorem 2.9, there are  $5^3 = 125$  distinct  $(1 + u^2)$ -constacyclic codes over  $R$  of length 7:

$$\mathcal{C}_{(l_1, l_2, l_3)} = \langle g_{(l_1, l_2, l_3)}(x) \rangle \pmod{x^7 - (1 + u^2)},$$

where  $g_{(l_1, l_2, l_3)}(x) = u^{l_1}e_1(x) + u^{l_2}e_2(x) + u^{l_3}e_3(x)$ ,  $0 \leq l_1, l_2, l_3 \leq 4$ . Moreover, the number of codewords contained in  $\mathcal{C}_{(l_1, l_2, l_3)}$  is equal to

$$|\mathcal{C}_{(l_1, l_2, l_3)}| = 2^{(4-l_1)+3(4-l_2)+3(4-l_3)} = 2^{28-(l_1+3(l_2+l_3))}.$$

- By Theorems 4.2, there are 5 distinct self-dual  $(1 + u^2)$ -constacyclic codes over  $R$  of length 7:

$$\mathcal{C}_{(2, l, 4-l)} = \langle g_{(2, l, 4-l)}(x) \rangle \pmod{x^7 - (1 + u^2)}, \quad 0 \leq l \leq 4,$$

where

$$\begin{aligned} g_{(2,0,4)}(x) &= u^2x^6 + u^2x^5 + (u^2 + 1)x^4 + u^2x^3 + (u^2 + 1)x^2 + x + u^2 + 1; \\ g_{(2,1,3)}(x) &= (u^3 + u^2)x^6 + (u^3 + u^2)x^5 + (u^2 + u)x^4 + (u^3 + u^2)x^3 + (u^2 + u)x^2 + (u^3 + u^2 + u)x + u^3 + u^2 + u; \\ g_{(2,2,2)}(x) &= u^2; \\ g_{(2,3,1)}(x) &= (u^2 + u)x^6 + (u^3 + u^2 + u)x^5 + (u^3 + u^2)x^4 + (u^3 + u^2)x^2 + (u^3 + u^2 + u)x^3 + (u^3 + u^2)x + u^3 + u^2 + u; \\ g_{(2,4,0)}(x) &= (u^2 + 1)x^6 + x^5 + x^4u^2 + x^3 + x^2u^2 + xu^2 + u^2 + 1. \end{aligned}$$

**Acknowledgments** Part of this work was done when Yonglin Cao was visiting Chern Institute of Mathematics, Nankai University, Tianjin, China. Yonglin Cao would like to thank the institution for the kind hospitality. This research is supported in part by the National Natural Science Foundation of China (Grant Nos. 11471255).

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